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# Helicity Theorem and Vortex Lines in Superfluid $^4$He

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The helicity conservation theorem is demonstrated in the case of superfluid $^4$He. As in the case of a classical barotropic fluid, the helicity integral expresses some topological properties of vortex lines.

## 1. INTRODUCTION

As noticed by Moreau (1961) and later by Moffat (1969), if **u** satisfies the Euler equations of an ideal barotropic fluid in $\mathbb{R}^3$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+\nabla p(\rho)=0$$

then the helicity integral (assuming that it exists)

$$I=\int_{\mathbb{R}^3}\mathbf{u}\cdot\operatorname{rot}\mathbf{u}\, d^3x \tag{1}$$

is a constant of motion. It expresses certain topological properties of the vortex lines. In the case of vortex filaments, the integral depends on whether or not the vortex lines are knotted or linked. If not, then $I=0$. Using the geometrical setting of superfluid equations, we formulate a more general version of the helicity theorem which is valid in the case of an ideal classical fluid and superfluid $^4$He as well.

## 2. EQUATIONS OF THE SUPERFLUID

To derive the flow equations, one may trace the movement of an "average particle" of the fluid. The Hamiltonian of such a particle can be

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taken as $H = m(\frac{1}{2}v^2 + \mu)$, where $\mu$ is the chemical potential. Similarly, one can define the action $S$

$$\frac{\partial S}{\partial x^\alpha} = A_\alpha = (-H, \mathbf{p}), \qquad \alpha = 0, 1, 2, 3$$

where $\mathbf{p}$ is the momentum of the particle. As is known, because of the appearance of quantized vortex lines in the superfluid, the action is not a univalued function of $t$ and $x$; it changes its value by $2\pi i$ when going around a single vortex line. Therefore it is more convenient to work with the action 1-form (for convenience divided by the atomic mass of $^4$He)

$$A = -(\mu + \tfrac{1}{2}v^2)\, dt + \mathbf{v} \cdot d\mathbf{x} \tag{2}$$

Note that $(m/h) \int_C A = n$, the number of quanta of circulation within contour $C$.

The second object which will be exploited here is the vorticity current 2-form, denoted as $J$, $J = J_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$. For any pair of vectors $X$, $Y \subset E^4$, we define $J(X, Y) = J_{\alpha\beta}(X^\alpha Y^\beta - Y^\alpha X^\beta)$ to be equal to $\hbar/m$ multiplied by the number of vortex lines crossing the surface element spanned on $X$, $Y$, taken with a plus sign if the orientation of the vortex line agrees with the orientation of the element and with a minus sign otherwise.

Since $A$ and $J$ are related by Stokes' theorem, they must satisfy the following equation:

$$dA = J \tag{3}$$

where $d$ denotes the exterior derivative. This is a strict consequence of the definition of $A$ and $J$.

For more detailed analysis of equation (3) we need to specify the structure of $J$. Let us define the Galilean four-component vortex line velocity as $V = (1, \mathbf{v}_L)$, $\mathbf{v}_L = (v_L^1, v_L^2, v_L^3)$.

The moving superfluid vortex lines are subjected to frictional forces from the side of the normal component and to the Magnus force appearing as a consequence of the relative velocity of the vortex line and the superfluid. Depending on the level of description and the physics that is involved, various approximations can be used to complete the equations of the superfluid in the presence of vortex lines. In one extreme case a microscopic description can be used in which one traces the motion of every vortex line, as it was done, for example, by Schwarz (1988) in order to justify the Vinen's equations of turbulent superfluid $^4$He. Another extreme case, useful for large densities of locally parallel vortex lines, consists in using the mean field description. In this case the vorticity, which is concentrated along the vortex filaments, is smeared out to obtain a continuous distribution. In this way one obtains the so-called HVBK equations (Hall and Vinen, 1956; Bekarevich and Khalatnikov, 1961).

According to Hall and Vinen, because of scattering, the rotons and phonons exert forces on a moving vortex line. In consequence, contrary to the case of an ordinary perfect fluid, the velocity of a quantized vortex line is different from the superfluid velocity. It moves in such a way that these extra forces are balanced by the Magnus force acting on the vortex line in the superfluid flow. Thus, in the presence of vortex lines there exists a sort of frictional force between both (superfluid and normal) fluid components. Consequently, the velocity of the line is assumed to be

$$\mathbf{v}_L = \tilde{\mathbf{v}}_s + \beta' \rho_s (\mathbf{v}_n - \tilde{\mathbf{v}}_s) + \beta \rho_s \mathbf{s} \times (\mathbf{v}_n - \tilde{\mathbf{v}}_s)$$

where $\mathbf{s} = \omega/|\omega|$ is the unit vector tangent to the vortex line, $\omega$ is the smeared-out superfluid vorticity, $\tilde{\mathbf{v}}_s = \mathbf{v}_s - \mathrm{rot}(\lambda s)$, $\mathbf{v}_s$ is the corresponding velocity of the superfluid, and $\mathbf{v}_n$ is the velocity of the normal fluid component. The coefficients $\beta$, $\beta'$, and $\lambda$ and the densities $\rho_s$ and $\rho_n$ are temperature dependent. The expression for $\mathbf{v}_L$ used in the HVBK theory corresponds to what is called the local induction approximation and used in the microscopic description (e.g., Schwarz, 1988). The fact that $\mathbf{v}_L \neq \mathbf{v}_s$ leads to the necessity of imposing some additional boundary conditions in order to have the uniqueness of the initial boundary problem (Peradzynski, 1988).

Let for any vector $X$ and any $k$-form, $X \lrcorner Q$ denote the $(k-1)$-form define by

$$(X \lrcorner Q)(Z_1, \ldots, Z_{k-1}) := Q(X, Z_1, \ldots, Z_{k-1})$$

The assumptions that the vortex lines are neither disappearing nor generated spontaneously in the fluid, except for the boundary, and that they move with velocity $\mathbf{v}_L$ result in the following relation:

$$\mathbf{V} \lrcorner J = 0 \tag{4}$$

In order to demonstrate this, let us take a small segment $\delta l$ at rest in the considered system of coordinates. As follows from the definition of $J$, within the time interval $\delta t$ the segment is crossed by $(J_{0\alpha} - J_{\alpha 0})\delta l^\alpha \delta t$ vortex lines. Obviously, when the segment moves together with the vortex lines, this quantity is equal to zero. This implies that in the system comoving with vortex lines, $(J_{0\alpha} - J_{\alpha 0})\delta l^\alpha$ is equal to zero for any $\delta l^\alpha$. In this system, however, $\mathbf{V} = (1, 0, 0, 0)$, $(\mathbf{V} \lrcorner J)(\delta l) = (J_{0\alpha} - J_{\alpha 0})\delta^\alpha l$, and therefore $\mathbf{V} \lrcorner J = 0$ in any system. As $J_{0\alpha} = 0$ (antisymmetry of $J_{\alpha\beta}$) in this comoving system, we have $J = \hat{\omega}$, where $\hat{\omega}$ is a purely spatial form, $\hat{\omega} = \hat{\omega}_{ab}\, dx^a \wedge dx^b$, $a, b = 1, 2, 3$. Transforming it back to the laboratory system $(dx^\alpha \to dx^\alpha - \mathbf{v}_L^\alpha\, dt)$, one arrives at

$$J = \hat{\omega} + (\mathbf{v}_L \lrcorner \hat{\omega}) \wedge dt \tag{5}$$

where $\mathbf{v}_L \lrcorner \hat{\omega} = \mathbf{v}_L^a(\hat{\omega}_{ab} - \hat{\omega}_{ba})\, dx^b$. The above representation allows one to associate $J$ with the vorticity field. Indeed, to any skew-symmetric 2-form $\hat{\omega}$ in $\mathbb{R}^3$ there corresponds a vector field $\omega$ ($\omega^a = \frac{1}{2}\varepsilon^{abc}\hat{\omega}_{bc}$, $\varepsilon$ an alternating symbol). One can rather easily verify the following lemmas (Peradzynski, 1988).

*Lemma 1.* The equation $dA = J$ is equivalent to the following set of equations:

$$\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s + \nabla \mu = -\omega \times (\mathbf{v}_L - \mathbf{v}_s) \tag{6}$$

$$\operatorname{rot} \mathbf{v}_s = \omega$$

Taking the exterior derivative of both sides of equation (3), one obtains

$$dJ = 0 \tag{7}$$

which is the conservation of vorticity. We have the following result.

*Lemma 2.* In a Cartesian system of coordinates, the equation $dJ = 0$ is equivalent to

$$\frac{\partial \omega}{\partial t} + \operatorname{rot}(\omega \times \mathbf{v}_L) = 0, \qquad \operatorname{div} \omega = 0 \tag{8}$$

The full set of equations must also contain the continuity equation, the equations for the normal fluid component, and the thermodynamic relations, which, however, will not be considered here.

Summarizing, the basic equations of the superfluid are

$$dA = J, \qquad J = \hat{\omega} + (\mathbf{v}_L \lrcorner \hat{\omega}) \wedge dt \tag{9}$$

while $dJ = 0$ is their consequence. Equations (9) are equivalent to

$$V \lrcorner dA = 0$$

which is the simplest way of writing the flow equations. In the vector notation it is equivalent to the first of equations (6) in which $\omega$ is replaced by $\operatorname{rot} \mathbf{v}_s$ and it expresses the fact that vortices (represented by $dA$) are traveling with velocity $V = (1, \mathbf{v}_L)$.

## 3. CONSERVATION OF HELICITY AND TOPOLOGY OF VORTEX LINES

Let us define the helicity current 3-form $\mathscr{H}$ in the space-time $E^4$,

$$\mathscr{H} := A \wedge dA$$

We have $d\mathcal{H} = dA \wedge dA = J \wedge J$. However, in the comoving system of coordinates, $J = \hat{\omega}$ and therefore $J \wedge J = \hat{\omega} \wedge \hat{\omega} = 0$. Thus, $J \wedge J$ vanishes in any systems of coordinates. Therefore

$$d\mathcal{H} = 0$$

and $\mathcal{H}$ represents a conserved current. Expressing this in terms of $\hat{\omega}$ and $\mathbf{v}_s$, one obtains

$$\mathcal{H} = -(\tfrac{1}{2}v_s^2 + \mu)\hat{\omega} \wedge dt + v_s \wedge (v_L \lrcorner \hat{\omega}) \wedge dt + v_s \wedge \hat{\omega}$$

By the natural duality between 3-forms and vectors (vector densities) in $E^4$, we may define the helicity flux vector field $\tilde{\mathcal{H}} := (\mathcal{H}^0, \boldsymbol{\mathcal{H}})$ where in a Cartesian system

$$\mathcal{H}^0 = \mathbf{v}_s \cdot \omega, \qquad \boldsymbol{\mathcal{H}} = +(\tfrac{1}{2}v_s^2 + \mu)\omega + \mathbf{v}_s \times (\mathbf{v}_L \times \omega)$$

Then the conservation equation $d\mathcal{H} = 0$ is equivalent to

$$\frac{\partial \mathcal{H}^0}{\partial t} + \operatorname{div} \boldsymbol{\mathcal{H}} = 0$$

The following theorem is an obvious consequence of our considerations.

*Theorem 1.* Let for any $t$, $dA$ be of compact support as a function of $x^1$, $x^2$, $x^3$, and let $t \to \Sigma_t$ be a foliation of hypersurfaces in $E^4$ parametrized by $t$; then $\int_{\Sigma_t} A \wedge dA$ does not depend on $t$.

Indeed, for any domain $\Omega$ with piecewise smooth boundary, in the space-time we have

$$\int_{\partial\Omega} A \wedge dA = \int_{\Omega} d\mathcal{H} = 0 \tag{10}$$

Taking for $\Omega$ the domain contained between hypersurfaces $\Sigma_0$ and $\Sigma_t$, one arrives at the thesis.

In particular, if $\Sigma_t$ are hypersurfaces $t = \text{const}$ in a Cartesian system of coordinates, then we have

$$\int_{\partial\Omega} A \wedge dA = \int_{\langle t=0\rangle} \mathbf{v}_s \cdot \operatorname{rot} \mathbf{v}_s \, d^3x - \int_{\langle t=t_0\rangle} \mathbf{v}_s \cdot \operatorname{rot} \mathbf{v}_s \, d^3x = 0 \tag{11}$$

which expresses the conservation of helicity in the form already known in classical fluid dynamics.

*Theorem 2.* Let $\Sigma$ be a bounded piece of a three-dimensional hypersurface. Then $\int_\Sigma A \wedge dA$ does not depend on the gauge transformations $A \to A + d\phi$ provided that either $\phi = 0$ on $\partial\Sigma$ or $d\phi \wedge A = 0$ on $\partial\Sigma$.

For the proof, notice that the following equalities are true:

$$\int_{\Sigma} d\phi \wedge dA = \int_{\Sigma} d(\phi\, dA) = \int_{\partial\Sigma} \phi\, dA = \int_{\partial\Sigma} A \wedge d\phi$$

In particular, $\int_{\mathbb{R}^3} \mathbf{v}_s \cdot \operatorname{rot} \mathbf{v}_s\, d^3x$, where $\operatorname{rot} \mathbf{v}_s$ is vanishing at $\infty$ (say, it is of compact support), does not depend on the gauge transformations $v \to v + \nabla\phi$. As was noticed by Moffat (1969), this integral is related to the "degree of knottedness" of vortex lines.

Suppose that vorticity is concentrated along a contour $C$. Any knotted contour can be transformed into a number of possibly linked simple circuits by inserting in certain places the pairs of segments of equal but opposite vorticity (Figure 1). For example, for a pair of linked contours $C'$, $C''$ in $\mathbb{R}^3$ the number

$$n = \frac{1}{4\pi} \oint_{C'} \oint_{C''} \frac{(\mathbf{x}' - \mathbf{x}'') \cdot (d\mathbf{l}'' \times d\mathbf{l}')}{|\mathbf{x}' - \mathbf{x}''|^3} \tag{12}$$

is an integer (Flanders, 1963) called the "winding" number of $C'$ and $C''$. The integral (12) corresponds to the helicity integral (11) for a special velocity field

$$v_0(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \mathbf{x}') \times \omega(x')}{|x - x'|^3}\, d^3\mathbf{x}' \tag{13}$$

where the vorticity $\omega(x')$ of unit strength is concentrated along $C'$ and $C''$ and therefore it is the same as was concentrated along $C$. Any solution $\mathbf{v}_s$ of $\operatorname{rot} \mathbf{v}_s = \omega$ with given $\omega$ can be transformed to the form (13) with the help of some gauge transformation. Moreover, in proving the helicity conservation theorem, we have not used any specific assumption about the vector field $\mathbf{v}_L$ which is responsible for the deformation of the vorticity field

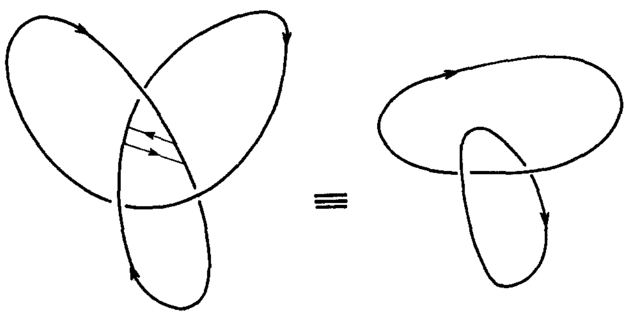

Fig. 1.

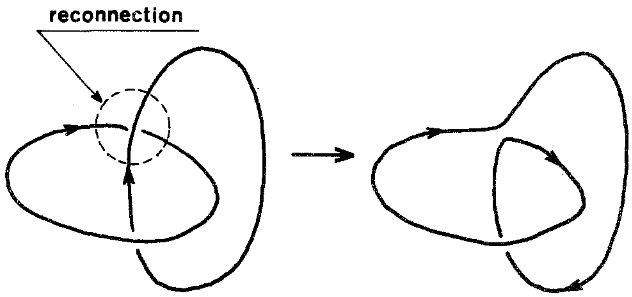


Fig. 2.

$\omega$. Therefore, the helicity integral does not depend on the specific field $\omega$; it depends on the topological properties of $\omega$.

In his approach to the superfluid turbulence, Schwarz (1978) proposed certain mechanism of reconnecting vortex lines when they cross each other. One may easily check that the reconnection mechanism proposed by Schwarz makes it possible to unknot the nontrivial knots or unlink the nontrivial links (Figure 2), thus violating the helicity conservation. This seems to suggest that the reconnection processes may lead to the appearance of extra forces (possibly dissipative) which were not taken into account in existing models.

Substitution of $\mathbf{v}_L = \mathbf{v}$ and $\mu = \int [dp/\rho(p)]$ converts the superfluid equations (3) into equations of the classical barotropic fluid. Although in this paper we considered the Galilean superfluid, one can easily generalize all the results to the relativistic case. This can be done by taking $A = \mu U_\alpha \, dx^\alpha$, where $\mu$ is the specific (per particle) chemical potential in the system at rest and $U_\alpha = g_{\alpha\beta} U^\beta$ is the covariant four-component velocity. Obviously, in order to obtain the structure of $J$ [see equation (9)] the Lorentz transformation must be used.

As in the Galilean case, the superfluid equations can be written as

$$V \lrcorner \, dA = 0$$

which implies that $dA \wedge dA$ represents the conserved helicity current. Clearly, to complete the flow equations, the appropriate (relativistic) expression for $V$ is needed. For a barotropic perfect fluid $V^\alpha = U^\alpha$.

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